A Kinematic Rejoin Method for Active Defense of Non-Maneuverable Aircraft

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***Abstract*— An engagement scenario involving the defense of a valuable and non-maneuverable aircraft is investigated in this paper. An Attacker missile pursuing a Target aircraft and a Defender missile which aims at intercepting the Attacker in order to protect the Target aircraft are considered. A zero-sum differential game is formulated where the objective of the Attacker is to minimize the terminal distance with respect to the Target aircraft and the objective of the Defender is to maximize the terminal Attacker-Target separation at the time of intercepting the Attacker. The saddle point state-feedback strategies for the Attacker and the Defender are obtained; these strategies are compared to other heuristic approaches. It is shown that better performance is obtained by implementation of the saddle point strategies derived in this paper.**

I. INTRODUCTION

Pursuit-evasion scenarios involving multiple agents represent important and challenging types of problems in aerospace, control, and robotics. They are also useful in order to analyze biologically inspired behaviors. For instance, the paper [1] addressed a scenario where two evaders employ coordinated strategies to evade a single pursuer, but also to keep them close to each other. The authors of [2] discussed a multi-player pursuit-evasion game with line segment obstacles labeled as the Prey, Protector, and Predator Game. A different approach to address pursuit-evasion games with several pursuers in order to capture an evader within a bounded domain is based on dynamic Voronoi diagrams, as in [3] and [4]. In [5] the Earliest Intercept Line (EIL) concept is described as a classic game theoretic framework used to create a geometric missile guidance strategy to optimize a value function through multiple missile guidance phases. In [6] a differential game with multiple attackers, multiple defenders, and a stationary target in a bounded domain is studied. The work in [7] provided a game formulation to solve reach and avoid problems involving nonlinear systems.

Further, the defense of static targets was addressed in [8] and [9]. Defense of aircraft when an interceptor uses a fixed guidance law was studied in [10]. This paper addresses the defense of non-maneuverable aircraft. The defense scenario which includes three agents, the Target ship, the Attacker missile, and the Defender missile (or counter-weapon) was first analyzed in refs. [11] and [12]. It was assumed that the Target holds a fixed course whereas the Attacker and the Defender use Collision Course (CC) guidance.

A different guidance law for the Target-Attacker-Defender (TAD) scenario was given in [13]. These authors investigated an interception method called Triangle Guidance (TG), where the objective is to command the defending missile to be on the line-of-sight between the attacking missile and the aircraft for all time while the aircraft follows some predetermined trajectory. The papers [14], [15] presented an analysis of the end-game TAD scenario based on the Attacker/Target miss distance for a *non-cooperative* Target/Defender. The authors develop linearization-based Attacker maneuvers in order to evade the Defender and continue pursuing the Target.

Li and Cruz [16] also considered the game of defending an asset from an attacking intruder using an interceptor, where the intruder and interceptor have the same speed. The authors considered different Target scenarios including the one when it follows a known trajectory. In [16], interception occurs when the Defender reaches a certain distance with respect to the Attacker; point capture is not enforced in that reference.

The work presented in this paper follows closely the recent reference [17] where a zero-sum differential game between the Attacker and the Defender is analyzed. Different from [16], the authors of [17] consider point capture and the case where the Defender is faster than the Attacker.

II. PREVIOUS WORK

In the defense differential game of non-maneuverable aircraft the speeds of the Target (T), Attacker (A), and Defender (D) are constant and they are denoted by v_T, v_A, and v_D, respectively. The agents have simple 2-D motion as it is commonly found in the games of Isaacs [8].

The complete state of the game is defined by $\mathbf{x} := (x_T, y_T, x_A, y_A, x_D, y_D) \in \mathbb{R}^6$. The Attacker's control variable is his instantaneous heading angle, $\mathbf{u}_A = \{\omega\}$. The Defender affects the state of the game by choosing the instantaneous heading ξ, so the Defender's control variable is $\mathbf{u}_B = \{\xi\}$. The dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_A, \mathbf{u}_B)$ are defined by the system of ordinary differential equations:

$$\dot{x}_T = v_T \cos\phi, \quad \dot{x}_A = v_A \cos\omega, \quad \dot{y}_D = v_D \sin\xi$$
$$\dot{y}_T = v_T \sin\phi, \quad \dot{y}_A = v_A \sin\omega, \quad \dot{x}_D = v_D \cos\xi \quad (1)$$

where the heading of the non-maneuverable Target aircraft, ϕ, is constant and it is assumed to be known to both players A and D. The admissible controls are given by $\omega, \xi \in [-\pi, \pi]$.

The defense differential game of non-maneuverable aircraft was addressed in the reference [17]. In that reference a heuristic approach was considered which is based on the solution of the game for defense of static targets previously studied by the same authors [9].

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The work in [9] revisited Isaacs' problem of guarding a static Target. In [8], p. 19, the Attacker and the Defender have same speed and simple motion. The Defender strives to intercept the Attacker in order to protect the static Target; its objective is to maximize the distance between Attacker and Target at the interception time t_f. The Attacker aims at minimizing the same distance. The optimal Attacker's strategy is to aim at the point on the orthogonal bisector of the segment $\overline{AD}$ which is the closest to the Target. The Attacker is intercepted by the Defender which aims at the same point. The optimal trajectories are straight lines.

The results in [9] extended the same problem for the case where the Defender is faster than the Attacker, $v_D > v_A$. The orthogonal bisector is then replaced by an Apollonius circle. Similarly, A aims at the point on the circle, where it is intercepted by D, which is the closest to the static Target. The solutions to the problems in [8] and [9] possess the invariance property, that is, *under optimal play*, the solution is recomputed at every time instant using the new positions of the players and the optimal interception point does not change.

The solution in [9] was heuristically applied to the defense differential game of non-maneuverable aircraft in reference [17]. At each point in time the players A and D compute a new Apollonius circle and a new aimpoint which is the point on the circle closest to the instantaneous Target position. The point is time-varying in general, hence, the invariance property is not preserved in this situation. An example of such approach is shown in Fig. 1, where the curved trajectories (blue lines) result by aiming at the time-varying interception point I.

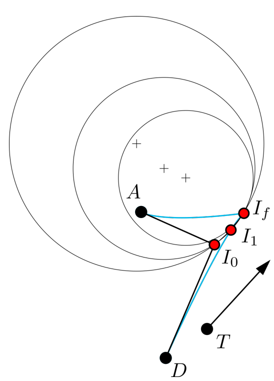

Fig. 1. Heuristic Approach

In this paper we formally analyze the defense differential game of non-maneuverable aircraft and provide the actual solution of the differential game by assuming that the heading of the non-maneuverable Target is known to the players A and D. This is not a restrictive assumption since the authors of [17] considered the defense of large aircraft, in such a case, the fixed heading of the Target can be quickly estimated by the players. This is in fact one of the main factors contributing to the success of Proportional Navigation guidance laws implemented by Attacker missiles against low maneuverable aircraft. [18]. Hence, from a practical point of view, there is no reason not to include the constant Target heading into the set of information available to the players A and D.

It is important to note that although the Target's heading is constant and it has no decision or control variable, its position coordinates (x_T, y_T) are considered in the state of the system since, as it will be shown below, the cost/payoff is a function of the Target's coordinates.

III. Differential Game

In this paper we will determine the saddle point strategies of the players A and D and we will show that these strategies are significantly different from the heuristic approach based on the static target solution which was implemented in [17]. In particular, we can show that if one of the players implements the optimal strategy derived in this paper while the opponent implements the heuristic approach in [17], then the first player will see the Value of the game shift in its favor, i.e. it benefits from the opponent playing a non-optimal, heuristic strategy.

Define the speed ratio problem parameter $\alpha = v_T/v_A$. In general, we have that the Attacker missile is faster than the Target aircraft, so $\alpha < 1$. Also, define the speed ratio $\beta = v_D/v_A$. When the Defender is faster than the Attacker we have that $\beta > 1$. The speed ratios α and β and the Target's constant heading ϕ are the problem parameters. The initial state of the system is defined as

$$\mathbf{x}_0 := (x_{T_0}, y_{T_0}, x_{A_0}, y_{A_0}, x_{D_0}, y_{D_0}) = \mathbf{x}(t_0).$$

In this paper, we confine our attention to point capture, that is, the $A-D$ separation has to become zero in order for the Defender to intercept the Attacker. We assume that given the problem parameters, the state of the system belongs to the escape set, denoted by $R_e \subset \mathbb{R}^6$. In other words, given the speed ratios, the Target heading, and $\mathbf{x}_0$, there exist an strategy for D to intercept A before the latter captures T. The game of capture, that is, the game when the state belongs to the complement of R_e is not addressed in this paper and will be investigated in future work.

The termination set which represents interception of the Attacker by the Defender (and the Target escapes) is then defined as follows:

$$\mathcal{C} := \left\{ \mathbf{x} \;|\sqrt{(x_A - x_D)^2 + (y_A - y_D)^2} = 0 \right\} \tag{2}$$

The terminal time t_f is defined as the time instant when the state of the system satisfies (2), at which time the terminal state is $\mathbf{x}_f := (x_{T_f}, y_{T_f}, x_{A_f}, y_{A_f}, x_{D_f}, y_{D_f}) = \mathbf{x}(t_f)$.

The terminal cost/payoff functional is

$$J(\mathbf{u}_A(t), \mathbf{u}_B(t); \mathbf{x}_0) = \Phi(\mathbf{x}_f) \tag{3}$$

where

$$\Phi(\mathbf{x}_f) := \sqrt{(x_{A_f} - x_{T_f})^2 + (y_{A_f} - y_{T_f})^2}. \tag{4}$$

The cost/payoff functional depends only on the terminal state - the active target defense differential game (ATDDG) is a terminal cost/Mayer type game. Its Value is given by:

$$V(\mathbf{x}_0) := \min_{\mathbf{u}_A(\cdot)} \max_{\mathbf{u}_B(\cdot)} J(\mathbf{u}_A(\cdot), \mathbf{u}_B(\cdot); \mathbf{x}_0) \tag{5}$$

subject to eq. (1)-(2), where $\mathbf{u}_A(\cdot)$ and $\mathbf{u}_B(\cdot)$ are the players' state feedback strategies.

The co-state is $\lambda^T = (\lambda_{x_A}, \lambda_{y_A}, \lambda_{x_D}, \lambda_{y_D}, \lambda_{x_T}, \lambda_{y_T}) \in \mathbb{R}^6$, and the Hamiltonian of the differential game is:

$$\begin{aligned}\mathcal{H} \ &= \lambda_{x_A}\cos\omega + \lambda_{y_A}\sin\omega + \beta\lambda_{x_D}\cos\xi \\ &\quad + \beta\lambda_{y_D}\sin\xi + \alpha\lambda_{x_T}\cos\phi + \alpha\lambda_{y_T}\sin\phi.\end{aligned} \tag{6}$$

where the speeds have been normalized using the Attacker's speed v_A.

Theorem 1: Consider the defense differential game of non-maneuverable aircraft eqn. (1)-(5). The headings of the Attacker and the Defender are constant under optimal play and their trajectories are straight lines.

IV. Main Results

Recall from Theorem 1 that the optimal trajectories are straight lines. In such an instance, the term Apollonius circle is a relevant tool to determine the optimal headings of the agents A and D. In general, a circle can be defined as the locus of points P with constant ratio of distances to two given points which are called foci. In our problem the foci are A and D, *i.e.*, $\gamma = \frac{\overline{AP}}{\overline{DP}}$ is constant. When the circle is defined using the constant ratio of distances just described, it is commonly referred to as an Apollonius circle and it represents an important tool to analyze pursuit-evasion problems. Consider agents A and D traveling in straight lines and at constant speeds v_A and v_D, respectively. The constant parameter $\gamma = \frac{v_A}{v_D} = \frac{1}{\beta}$ is the speed ratio parameter. In this scenario D strives to intercept A. D intercepts A at a point $I = (x_I, y_I)$ on the Apollonius circle and at that point the distance traveled by A is equal to γ times the distance traveled by D. Hence, an Apollonius circle can be constructed based on the distance between A and D and also based on the speed ratio parameter γ. The center of the circle is denoted by O where the points A, D, and O are collinear as it is illustrated in Fig. 2.

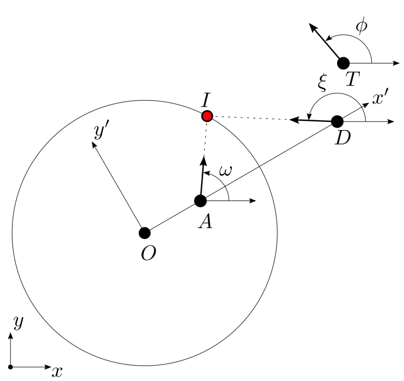

Fig. 2. Apollonius Circle (A,D) with Interception in Global Frame

Without loss of generality, we consider the relative coordinate frame illustrated in Fig. 3 where the points A and D denote the positions of the Attacker and the Defender, respectively. The x'-axis of this frame goes from A to D which results in $y_A = y_D = 0$. The origin of the coordinate frame is the center of the $\overline{AD}$-Apollonius. This circle is characterized as follows: Let

$$\lambda_A^D = \arctan\big((y_D - y_A)/(x_D - x_A)\big) \tag{7}$$

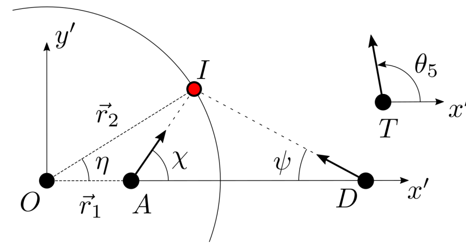

Fig. 3. Apollonius Circle (A,D) with Interception in Local Frame

be the line of sight (LOS) angle from A to D. The headings of the players in the relative frame are given by $\chi = \omega - \lambda_A^D$ and $\psi = \pi + \lambda_A^D - \xi$. The constant heading of the Target is $\theta_5 = \phi - \lambda_A^D$.

Let r_1 denote the distance between A, the Attacker position, and O, the center of the Apollonius circle. The distance r_1 is given by

$$r_1 = (\gamma^2 d)/(1-\gamma^2) \tag{8}$$

where $d = \sqrt{(x_A - x_D)^2 + (y_A - y_D)^2}$ is the distance between agents A and D. Also, let r_2 be the radius of the Apollonius circle. Then, r_2 is given by

$$r_2 = \gamma d/(1-\gamma^2). \tag{9}$$

The Attacker strives to minimize the terminal distance between itself and the Target which is traveling both at constant speed v_T and at constant heading ϕ. The points T and T' represent the initial and terminal positions of the Target, respectively. The point I is the final position of the Attacker and also the final position of the Defender since point capture is addressed.

Using the Apollonius circle between A and D, the problem can be transformed into an optimization problem in one variable. The Attacker aims at a point on the Apollonius circle, where it will be intercepted by the Defender, which minimizes the terminal $\overline{AT}$ distance. The optimal interception point is calculated as follows.

Theorem 2: Assume that $\mathbf{x} \in R_e$, then, the optimal interception point in the relative coordinate frame is given by $I^* = (r_2\cos\eta^*, r_2\sin\eta^*)$ where η^* is such that $\nu^* = e^{i\eta^*} = \cos\eta^* + i\sin\eta^*$ and ν^* is the solution of the polynomial equation

$$\begin{aligned}&(\tfrac{3r_1r_2}{2p})^2\nu^8 + b_7\nu^7 + b_6\nu^6 + b_5\nu^5 + b_4\nu^4 \\ &+ b_3\nu^3 + b_2\nu^2 + b_1\nu + (\tfrac{3}{2}r_1r_2p)^2 = 0\end{aligned} \tag{10}$$

which minimizes the function

$$\begin{aligned}J(\nu) =& r_2^2 + r_7^2 + \alpha^2(r_1^2 + r_2^2) \\ &- r_2(\alpha^2 r_1 + r_7 m^{-1})\nu - r_2(\alpha^2 r_1 + r_7 m)\nu^{-1} \\ &+ \alpha[r_7 Q - r_2(p^{-1}\nu + p\nu^{-1})]r_3(\nu)\end{aligned} \tag{11}$$

where

$$r_3(\nu) = \sqrt{r_1^2 + r_2^2 - r_1 r_2(\nu + \nu^{-1})}, \tag{12}$$

and $\nu = e^{i\eta}$. The polynomial coefficients are

$$\begin{aligned}
b_7 &= r_1 r_2[(\alpha r_1 + \tfrac{r_7}{\alpha m})^2 - \tfrac{3}{p}(\tfrac{r_1 r_7 Q}{2} + \tfrac{r_1^2 + r_2^2}{p})] \\
b_6 &= (\tfrac{r_1 r_7 Q}{2} + \tfrac{r_1^2+r_2^2}{p})^2 - \tfrac{3}{2} r_1^2 r_2^2 (1 - \tfrac{1}{p^2}) \\
&\quad -(r_1^2 + r_2^2)(\alpha r_1 + \tfrac{r_7}{\alpha m})^2 \\
b_5 &= r_1 r_2[(r_1^2 + r_2^2)(4 - \tfrac{1}{p^2}) + r_1 r_7 Q(\tfrac{1}{p} + \tfrac{1}{2}p) - \alpha^2 r_1^2 \\
&\quad -2 r_1 r_7 m + \tfrac{r_7^2}{\alpha^2}(\tfrac{1}{m^2} - 2)] \\
b_4 &= \tfrac{r_1^2 r_2^2}{4}(p^2 + \tfrac{1}{p^2} - 20) \\
&\quad +2(r_1^2 + r_2^2)(\alpha^2 r_1^2 + r_1 r_7 (m + \tfrac{1}{m}) + \tfrac{r_7^2}{\alpha^2}) \\
&\quad -2(\tfrac{r_1 r_7 Q}{2} + \tfrac{r_1^2 + r_2^2}{p})(\tfrac{r_1 r_7 Q}{2} + p(r_1^2 + r_2^2)) \\
b_3 &= r_1 r_2[(r_1^2 + r_2^2)(4 - p^2) + r_1 r_7 Q(p + \tfrac{1}{2p}) - \alpha^2 r_1^2 \\
&\quad -\tfrac{2}{m} r_1 r_7 + \tfrac{r_7^2}{\alpha^2}(m^2 - 2)] \\
b_2 &= (\tfrac{r_1 r_7 Q}{2} + p(r_1^2 + r_2^2))^2 + \tfrac{3}{2} r_1^2 r_2^2 (p^2 - 1) \\
&\quad -(r_1^2 + r_2^2)(\alpha r_1 + \tfrac{r_7 m}{\alpha})^2 \\
b_1 &= r_1 r_2[(\alpha r_1 + \tfrac{r_7 m}{\alpha})^2 - 3p(\tfrac{r_1 r_7 Q}{2} + p(r_1^2 + r_2^2))]
\end{aligned} \quad (13)$$

where the parameters $p = e^{i\theta_5}$, $m = e^{i\theta_7}$, and $Q = pm^{-1} + p^{-1}m$.

Proof. Considering the geometry in Fig. 3, one may be able to derive a kinematic linkage which represents the engagement scenario. The Apollonius circle can be represented by a crank slider mechanism as seen in Fig. 4. This geometry can be described as a linkage of fixed length $r_2 = \overline{OA'}$ that rotates about the origin O. As this linkage rotates about the origin, the linkage length $r_3 = \overline{AA'}$ can vary; however, the linkage is anchored and free to rotate about point A which is co-linear with the x'-axis, i.e. $\overline{OA}$ lies on the x'-axis. In this initial problem statement, the lengths r_1 and r_2 are fixed. r_3 is free to change in length and to rotate about A. One can visualize this motion quite easily, and the following kinematic synthesis shows the relation between the rational angle η, the length of the slider, r_3, and its angle with the x'-axis, χ.

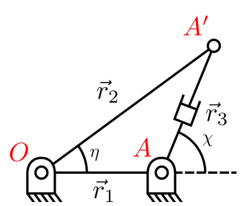

Fig. 4. Apollonius Circle as Linkage Geometry

The first step in solving the relation between the free variable of the linkage is to start with the closure equation:

$$\vec{r}_2 = \vec{r}_1 + \vec{r}_3 \quad (14)$$

By expanding the closure equation to include the Cartesian unit vectors: $\hat{i}$ and $\hat{j}$:

$$|\vec{r}_2|\cos(\eta)\hat{i} + |\vec{r}_2|\sin(\eta)\hat{j} = |\vec{r}_1|\hat{i} + |\vec{r}_3|\cos(\chi)\hat{i} + |\vec{r}_3|\sin(\chi)\hat{j} \quad (15)$$

Defining the shorthand notation:

$$\begin{aligned}
|\vec{r}_i| &= r_i \; \forall \; i = \{1, 2, \cdots, n\} \\
\cos(\eta) = c_\eta \quad &\& \quad \sin(\eta) = s_\eta \\
\cos(\chi) = c_\chi \quad &\& \quad \sin(\chi) = s_\chi
\end{aligned} \quad (16)$$

Using this shorthand, we can more compactly write the expanded closure equation as:

$$r_2 c_\eta \hat{i} + r_2 s_\eta \hat{j} = r_1 \hat{i} + r_3 c_\chi \hat{i} + r_3 s_\chi \hat{j} \quad (17)$$

Breaking eq. (17) into the x and y components and by isolating the slider, r_3, to the left hand side, squaring both sides, then summing both equations, we eliminate the unknown angle χ from the equation. This allows us to solve for the slider length as a function of η:

$$r_3^2 = r_2^2 + r_1^2 - 2 r_1 r_2 c_\eta \quad (18)$$

Now that we have the slider lengths in terms of known variables, we can then express the relation of the angles χ and η using eq. (17): $s_\chi = r_2 s_\eta / r_3$.

Next we consider the entire kinematic rejoin geometry as seen in Fig. 5.

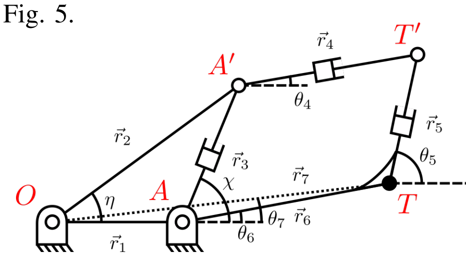

Fig. 5. Kinematic Geometry

Similar to the previous analysis we begin with the loop closure equation:

$$\vec{r}_2 + \vec{r}_4 = \vec{r}_7 + \vec{r}_5 \quad (19)$$

Expanding eq. (19) into x and y Cartesian unit vectors and seperating them into component equations we obtain:

$$r_2 c_\eta + r_4 c_4 = r_7 c_7 + r_5 c_5 \quad (20)$$

$$r_2 s_\eta + r_4 s_4 = r_7 s_7 + r_5 s_5 \quad (21)$$

Since we are interested in the angle η that minimizes the distance r_4, we need to eliminate the unknown angle θ_4 from the component equations. Much like before, we can accomplish this by isolating r_4 on the left hand side of the equals sign. Squaring both sides and summing the two resulting equations:

$$\begin{aligned}
r_4^2 &= r_7^2 + r_5^2 + r_2^2 - 2 r_2 r_5 (c_\eta c_5 + s_\eta s_5) \\
&\quad - 2 r_2 r_7 (c_\eta c_7 + s_\eta s_7) + 2 r_5 r_7 (c_5 c_7 + s_5 s_7)
\end{aligned} \quad (22)$$

Next, we substitute the relationship between the linkage lengths r_5 and r_3. For this problem, we recall r_5 and r_3 have a linear relationship by a positive definite constant α, such that $r_5 = \alpha r_3$. Using this relation in eq. (22) yields:

$$\begin{aligned}
r_4^2 &= r_2^2 + \alpha^2 r_3^2 + r_7^2 - 2\alpha r_2 r_3 (c_\eta c_5 + s_\eta s_5) \\
&\quad - 2 r_2 r_7 (c_\eta c_7 + s_\eta s_7) + 2\alpha r_3 r_7 (c_5 c_7 + s_5 s_7)
\end{aligned} \quad (23)$$

Since it is our goal to minimize the linkage length r_4 we recognize that the minimization of r_4 is the same as a minimization of r_4^2. To perform the minimization, the cost/payoff can be written as follows:

$$\begin{aligned}
J &= r_2^2 + r_7^2 + \alpha^2 r_3^2 - 2\alpha r_2 \cos(\eta - \theta_5) r_3 \\
&\quad - 2 r_2 r_7 \cos(\eta - \theta_7) + 2\alpha r_7 \cos(\theta_5 - \theta_7) r_3
\end{aligned} \quad (24)$$

where $r_2, r_7, \theta_5, \theta_7$ are constant and only r_3 is a function of the angle η.

$$r_3 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\eta}. \tag{25}$$

From eq. (25) and the identity $\cos\eta = \frac{1}{2}(e^{i\eta} + e^{-i\eta})$, we can write r_3 in terms of $\nu = e^{i\eta} = \cos\eta + i\sin\eta$ as it is shown in eq. (12). Further, eq. (24) can be written as a function of ν in the following form:

$$\begin{aligned} J =& r_2^2 + r_7^2 + \alpha^2(r_1^2 + r_2^2) \\ &-r_2(\alpha^2 r_1 + r_7 m^{-1})\nu - r_2(\alpha^2 r_1 + r_7 m)\nu^{-1} \\ &+\alpha[r_7 Q - r_2(p^{-1}\nu + p\nu^{-1})]\sqrt{r_1^2 + r_2^2 - r_1r_2(\nu + \nu^{-1})} \end{aligned}$$

Differentiating yeilds:

$$\begin{aligned} \frac{dJ}{d\nu} =& -r_2(\alpha^2 r_1 + r_7 m^{-1}) + r_2(\alpha^2 r_1 + r_7 m)\nu^{-2} \\ &+ \alpha[r_7 Q - r_2(p^{-1}\nu + p\nu^{-1})]\frac{r_1 r_2(\nu^{-2} - 1)}{2\sqrt{r_1^2 + r_2^2 - r_1r_2(\nu + \nu^{-1})}} \\ &+ \alpha r_2\sqrt{r_1^2 + r_2^2 - r_1r_2(\nu + \nu^{-1})}(-p^{-1} + p\nu^{-2}) \end{aligned} \tag{26}$$

Setting eq. (26) equal to zero and multiplying by r_3 we obtain the following:

$$\begin{aligned} &[(\alpha^2 r_1 + r_7 m)\nu^{-2} - (\alpha^2 r_1 + r_7 m^{-1})] \\ &\quad\times\sqrt{r_1^2 + r_2^2 - r_1r_2(\nu + \nu^{-1})} \\ &+\frac{\alpha}{2}r_1(\nu^{-2} - 1)[r_7 Q - r_2(p^{-1}\nu + p\nu^{-1})] \\ &+\alpha(p\nu^{-2} - p^{-1})[r_1^2 + r_2^2 - r_1r_2(\nu + \nu^{-1})] = 0 \end{aligned} \tag{27}$$

We now divide by α and move the terms containing r_3 to the right hand side of eq. (27). In order to cancel negative exponents on ν we also multiply both sides of the equation by ν^3. After applying these operations we obtain:

$$\begin{aligned} &\frac{r_1}{2}(1 - \nu^2)[r_7 Q\nu - r_2(p^{-1}\nu^2 + p)] \\ &+(p - p^{-1}\nu^2)[(r_1^2 + r_2^2)\nu - r_1r_2(\nu^2 + 1)] = \\ &-\frac{\alpha^2 r_1 + r_7 m - (\alpha^2 r_1 + r_7 m^{-1})\nu^2}{\alpha}\sqrt{(r_1^2 + r_2^2)\nu^2 - r_1r_2(\nu^3 + \nu)} \end{aligned} \tag{28}$$

We can now take the square of both sides of eq. (28) and arrange common terms in order to obtain eq. (10). □

Remark. The solution provided in Theorem 2 only requires the rooting of a polynomial and determining the optimal solution, $\nu^* = \cos\eta^* + i\sin\eta^*$, by computing the associated cost of each root. The angle η^*, is uniquely determined from $\eta^* = \arccos Re(\nu^*)$ and $\eta^* = \arcsin Im(\nu^*)$, where $Re(\nu^*)$ and $Im(\nu^*)$ represent the real and the imaginary part of ν^*. This solution, although not explicit, can be easily implemented in state-feedback form which is useful to provide robustness against unknown guidance laws by the Attacker or different interception strategies by the Defender. In other words, given the state-feedback solution, anyone of the players will see its performance level increased if the opponent does not follow the optimal strategy obtained in Theorem 2.

A. Particular case $\gamma = 1$

In the case where A and D have the same speed, $\gamma = \frac{v_A}{v_D} = 1$, the optimal interception point can be obtained by rooting a quartic equation. Consider in this case the relative frame shown in Fig. 6 where the reachable regions of A and D are separated by the orthogonal bisector of the segment $\overline{AD}$ instead of the Apollonius circle previously described. Therefore, the optimal interception point in this case has coordinates $I^* = (0, y^*)$ where y^* is obtained as follows.

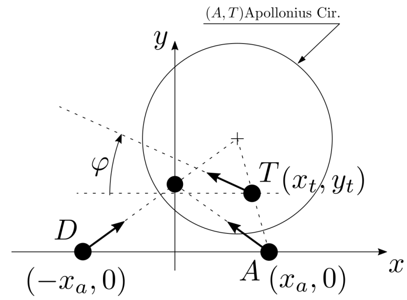

Fig. 6. Apollonius Circle for Particular Case $\gamma = 1$

Corollary 1: Consider the case $\gamma = 1$ and assume that $\mathbf{x} \in R_e$, then, the optimal interception point in the relative coordinate frame is given by $I^* = (0, y^*)$ where y^* the solution of the polynomial equation

$$\begin{aligned} &(1 + \alpha^2)^2 y^4 - 2(1 + \alpha^2)y_T y^3 \\ &+[y_T^2 + (1 + \alpha^2)x_D^2 - \alpha^2(y_T\sin\varphi - x_T\cos\varphi)^2]y^2 \\ &-2x_D^2[(1 + \alpha^2\cos^2\varphi)y_T - \alpha^2 x_T\sin\varphi\cos\varphi]y \\ &+x_D^2(y_T^2 - x_D^2\sin^2\varphi) = 0 \end{aligned} \tag{29}$$

which minimizes the function

$$\begin{aligned} J(y) =& (x_T - \alpha\sqrt{x_D^2 + y^2}\cos\varphi)^2 \\ &+ (y_T - y + \alpha\sqrt{x_D^2 + y^2}\sin\varphi)^2 \end{aligned} \tag{30}$$

V. EXAMPLES

Consider the same example as in [17], Section III.B. The initial positions are $D_0 = (0, 0)$, $A_0 = (0, 10)$, $T_0 = (5, 5)$. The speeds of the missiles are $v_D = 2$, $v_A = 1$. We also use the same Target's speed as in [17], $v_T = 1$. The Target fixed heading is $\phi = 60\,\text{deg}$.

Example 1. Fig. 7 shows the optimal trajectories of the encounter, where each one of the players, A and D, implement the saddle point strategies which were obtained in Section IV. The Value of the Game is $V(\mathbf{x}; \omega^*, \xi^*) = 1.9864$.

Note that the optimal strategies are continuously updated. This means that at every time instant, the current positions are used to update the state of the system and compute the optimal interception point which is used to obtain the headings of A and D. The solution of the defense differential game of non-maneuverable aircraft possess the invariance property, that is, the interception point and the headings of the players are constant under optimal play. The interception point in the fixed frame remains the same when both players apply the optimal strategy derived in this paper. The previous statement does not hold if at least one of the players does not follow it prescribed optimal strategy. An illustrative comparison with respect to the heuristic approach in [17] is shown in the next two examples. It is also shown that the player which implements the heuristic approach loses performance with respect to the objective of the game.

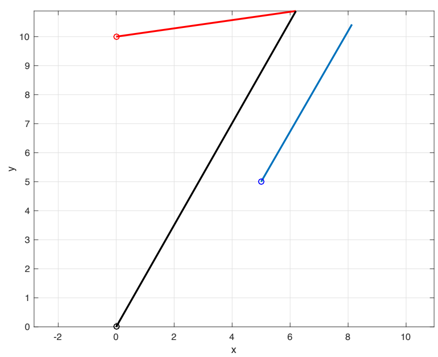

Fig. 7. Trajectories in Fixed Frame: A Optimal, D Optimal

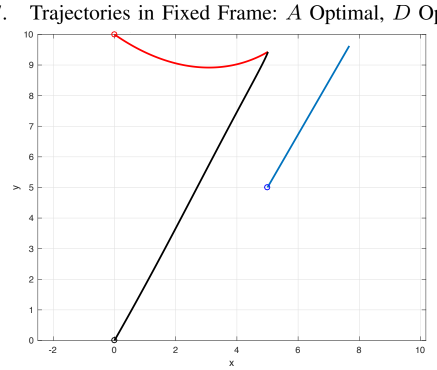

Fig. 8. Trajectories in Fixed Frame: A Heuristic, D Optimal

Example 2. Now consider the case where the Attacker does not follow its optimal strategy derived in this paper. Instead, it follows the heuristic approach in [17]. The Defender implements the optimal strategy obtained in this paper. The trajectories of the engagement under this selection of headings are shown in Fig. 8. The Defender is not only able to intercept the Attacker but it does at a distance further apart from the Target. The terminal $\overline{AT}$ separation is $V(\mathbf{x};\omega,\xi^*) = 2.6613$, that is, the Attacker is doing poorly by following a heuristic approach compared to the optimal saddle point strategy in Example 1.

Example 3. Finally, consider the opposite case, that is, the Attacker follows the optimal strategy derived in this paper while the Defender implements the heuristic approach in [17]. The resulting trajectories are shown in Fig. 9. The terminal $\overline{AT}$ separation is $V(\mathbf{x};\omega^*,\xi) = 1.9795$. In this case the terminal distance is less than the value obtained from the saddle point solution in Example 1 and the Defender's performance is deteriorated by following the heuristic approach. From this simulations analysis we have that $V(\mathbf{x};\omega^*,\xi) \leq V(\mathbf{x};\omega^*,\xi^*) \leq V(\mathbf{x};\omega,\xi^*)$, that is, the saddle point property holds.

VI. Conclusions

In Conclusion, using the zero-sum differential game formulation, the saddle point state-feedback strategies for the Attacker and the Defender were obtained using the Hamiltonian of the differential game. Further, we proved, using kinematics, the optimal interception point in the relative coordinate frames. We also showed, that any deviation from this optimal solution has poorer performance as described by three examples. This closed form solution, not only outperforms heuristic approaches, which was the aim of this paper, but can be solved using kinematics.

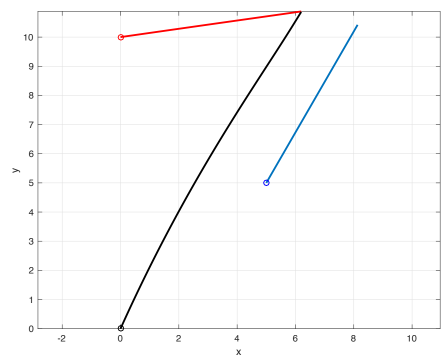

Fig. 9. Trajectories in Fixed Frame: A Optimal, D Heuristic

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